Density Contrast–Peculiar Velocity Relation in the Newtonian Gauge

Sohrab Rahvar*

Department of Physics, Sharif University of Technology,
P.O.Box 11365–9161, Tehran, Iran

and

Institute for Studies in Theoretical Physics and Mathematics,
P.O.Box 19395–5531, Tehran, Iran

February 1, 2008

Abstract

In general relativistic framework of large scale structure formation theory in the universe, we investigate relation between density contrast and peculiar velocity in the Newtonian gauge. According to the gauge–invariant property of the energy–momentum tensor in the this gauge, the velocity perturbation behaves as the Newtonian peculiar velocity. In this framework, relation between peculiar velocity and density contrast with respect to the Newtonian Peebles formula has an extra correction term which is ignorable for the small scales structures. The relativistic correction of peculiar velocity for the structures with the extension of hundred mega parsec is about few percent which is smaller than the accuracy of the recent peculiar velocity measurements. We also study CMB anisotropies due to the Doppler effect in the Newtonian gauge comparing with using the Newtonian gravity.

1   Introduction

In the large structure formation theory, structures originate from small density fluctuations that are amplified by gravitational instability. These initial fluctuations are assumed to be generated during inflationary epoch and are inflated to the beyond of the horizon while the size of horizon remains constant during the inflation. These quantum fluctuations beyond the horizon freeze and evolve like classical perturbations. They reenter the horizon at a later time when the horizon grows to the size of perturbation. The observational evidence for these small density fluctuations can be seen at the decoupling epoch as the anisotropy of CMB and also existence of the large scale structures in the universe. One of the consequences of density fluctuation is metric perturbations in the Friedman–Robertson–Walker (FRW) universe and deviation of velocity field from the Hubble law which is called *peculiar velocity*. Hence studying peculiar velocity could be a useful indirect

---

*E-mail:rahvar@theory.ipm.ac.ir

1

method to find out the structures in the FRW universe. In the Newtonian linear structure formation theory, relation between density contrast and peculiar velocity has been obtained by Peebles [1]. From the experimental point of view, Bertschinger and Dekel introduced POTENT method to reconstruct velocity potential from the radial component of velocity field [2]. Recently Branchini *et al* combined measurement of radial velocity of galaxies and independent measurement of density contrast to estimate Ω [3].

On the scales well below than the Hubble radius, the Newtonian theory of gravitation is a good approximation and Peebles' approach is widely used. However for observations and simulations are probing scales which are a significant fraction of the Hubble radius, the light-cone effect should taken into account [4]. Our aim in this work is to generalize relation between density contrast and peculiar velocity to general relativistic framework. The relativistic linear perturbation of FRW universe can be studied by two type of formulations exist in the literature. The 'gauge fixing' formulation, originated by Lifshitz [5], considers perturbed components of the metric which are related to the energy momentum tensor. According to the gauge freedom problem in the theory, it is necessary to put a constraint on perturbed part of metric for gauge fixing (e.g. synchronous gauge). In the Second approach, initiated by Bardeen in 1980 [6] gauge invariant variables can be made by the combining perturbed metric elements. These gauge-invariant variables are well known from other physical theories. For example, in classical electrodynamics it is usually more physical to work in terms of gauge-invariant electromagnetic fields rather than in terms of gauge-dependent scalar and vector potentials. The subsequent works in FRW perturbation theory can be found in some textbooks [7]-[10].

General relativistic analysis of peculiar velocity has been studied in the 'covariant' fluid flow [11], 'Harmonic gauge' [4] and ' quasi-Newtonian gauge' fixing methods. In this work we are going to use conformal Newtonian gauge also known as the (Longitudinal gauge) to obtain relation between density contrast and peculiar velocity. The conformal Newtonian gauge was introduced by Mukhanov *et al* [12] as a simple gauge used for scalar mode of metric perturbation. In this gauge perturbation of metric elements are gauge-invariant variables. For the case of absence of non-diagonal space–space component in energy–momentum tensor, perturbation of metric can be interpreted as relativistic generalization of Newtonian gravitational potential. We generalize this idea for peculiar velocity field according to its gauge–invariant property in the Energy–Momentum tensor. Here, we restrict ourselves to the flat universe, in the matter and radiation dominant epochs, and obtain relativistic relation between density contrast and peculiar velocity field. It is shown that in small scales compare to the size of horizon, where general relativistic effect due to the light cone effect is negligible, our formulation reduces to Peebles equation. For the structures with $300Mpc$ extension, the relativistic correction is about one percent. We also calculate the relativistic Doppler effect contribution on the CMB anisotropies and compare it with the Newtonian one.

The organization of the paper is as follows. In Sec. II we write down perturbation of metric in Newtonian gauge. In Sec. III we introduce energy–momentum tensor and Einstein equations in this gauge and In Sec. IV, we obtain relativistic relation between density contrast and peculiar velocity for flat universe in radiation and matter dominant epochs. In Sec. V , we obtain signature of Doppler effect on the anisotropies of CMB, using relativistic peculiar velocity. We conclude in Sec. VI with a brief summary and some discussions.

2 Perturbation of FRW in the Newtonian Gauge

Consider a small perturbation of metric $h_{\mu\nu}$ with respect to FRW background $g_{\mu\nu}$

$$ds^2 = (g_{\mu\nu} + h_{\mu\nu})dx^\mu dx^\nu \tag{1}$$

where $g_{\mu\nu} = a(\tau)^2(d\tau^2 - \gamma_{ij}dx^i dx^j)$ and τ is conformal time. Greek and Roman letters go from 0 to 3 and 1 to 3, respectively. The metric perturbation $h_{\mu\nu}$ could be categorized into three distinct types like Scalar, vector and tensor perturbation. This classification refers to the way in which $h_{\mu\nu}$ transforms under three–space coordinate transformation on the constant time hyper-surface. A covariant description of tensor decomposition has been shown in [13]. Vector and tensor modes exhibit no instability. Vector perturbation decays kinematically in an expanding universe and tensor perturbation leads to gravitational waves which do not couple to energy density and pressure inhomogeneities. However, scalar perturbation may lead to growing inhomogeneities of matter.

The most general form of the scalar metric perturbation is constructed by using four scalar quantities ϕ, ψ, B and E:

$$h_{\mu\nu}^{(s)} = \begin{pmatrix} 2\phi & -B_{;i} \\ B_{;i} & 2(\psi\gamma_{ij} - E_{;ij}) \end{pmatrix}. \tag{2}$$

One of the main difficulties of the relativistic structure formation theory is that there is gauge freedom in the theory and one can make an artificial perturbation by coordinate transformation in such a way that infinitesimal space–time distance between the two event remains constant. One way to overcome this problem is using gauge fixing and the other way is gauge invariant method [6]. In the latter method, some quantities are found to be gauge invariant under coordinate transformation. It seems that the gauge–invariant quantities are similar to the electric and magnetic fields in the theory of electrodynamics where they are measurable quantities in contrast to the vector and scalar potentials that are gauge dependent parameters. The simplest gauge–invariant quantities from the linear combination of ϕ, ψ, B and E which span the two–dimension space of gauge–invariant variables are:

$$\Phi^{(gi)} = \phi + \frac{1}{a}[(B - E')a]' \tag{3}$$

$$\Psi^{(gi)} = \psi - \frac{a'}{a}(B - E') \tag{4}$$

where prime represents derivation with respect to conformal time and index *(gi)* stands for gauge invariance. The above variables were first introduced by Bardeen [6]. Newtonian gauge is defined by choosing $B = E = 0$ [12]. It is seen that in this gauge ϕ and ψ become gauge–invariant variables and the metric can be written as follows:

$$ds^2 = a(\tau)^2 \left[(1 + 2\phi)d\tau^2 - (1 - 2\psi)\gamma_{ij}dx^i dx^j \right]. \tag{5}$$

3 Einstein Equation in The Newtonian Gauge

In the perturbed background of the FRW metric, the linear perturbation of Einstein equation can be written as follows:

$$\delta G^\mu_{\ \nu} = 8\pi G \delta T^\mu_{\ \nu}. \tag{6}$$

The left hand side of eq.(6) can be obtained from perturbation of metric in the Newtonian gauge and the right hand side of it is obtained by perturbed energy–momentum tensor. Restricting our attention to scalar perturbation of energy–momentum tensor, we can express the most general first order energy–momentum tensor in terms of four scalars as follows [6]:

$$\delta T^{\mu}_{\ \nu} = \begin{pmatrix} \delta\rho & -(\rho+p)a^{-1}u_i \\ (\rho+p)au_i & -\delta p + \sigma_{;ij} \end{pmatrix}, \tag{7}$$

where $\delta\rho$ and δp are the perturbations of energy density and pressure, $\mathcal{W}$ is potential for the three–velocity field $v^i = a(t)u^i(x,\tau) = \nabla^i\mathcal{W}$ and σ is anisotropic stress. Bardeen has shown that in the Newtonian gauge: $\delta T^{\mu(gi)}_{\ \nu} = \delta T^{\mu}_{\ \nu}$ [6]. According to this result, $v^i = a(t)u^i$ may be regarded as Newtonian peculiar velocity. For a perfect fluid, the anisotropic stress which leads to non-diagonal space–space component of the energy–momentum tensor vanishes. In this case it can be shown that $\phi = \psi$. Using the metric of eq.(5) in the left hand side and eq.(7) in the right hand side of eq.(6), Einstein equation can be rewritten:

$$\nabla^2\phi - 3\mathcal{H}\phi' - 3(\mathcal{H}^2 - \kappa)\phi = 4\phi G\delta\rho, \tag{8}$$

$$(a\phi)'_{,i} = -4\phi G(\rho+p)u_i, \tag{9}$$

$$\phi'' + 3\mathcal{H}\phi' + (2\mathcal{H}' + \mathcal{H}^2 - \kappa)\phi = 4\phi G\delta p, \tag{10}$$

where $\mathcal{H}$ is the Hubble constant in conformal time. For simplicity in calculation we consider the following change of variable.

$$\phi = 4\pi G(\rho+p)^{1/2}\omega = (4\pi G)^{1/2}[(\mathcal{H}^2 - \mathcal{H}' + \kappa)a^2]^{1/2}\omega. \tag{11}$$

By substituting eq.(11) into eq.(10), this equation is rewritten in terms of ω as follows:

$$\omega'' - c_s^2\nabla^2\omega - \frac{\theta''}{\theta}\omega = 0, \tag{12}$$

where

$$\theta = \frac{1}{a}\left(\frac{\rho}{\rho+p}\right)^{1/2}\left(1 - \frac{3\kappa}{8\pi G\rho a^2}\right)^{1/2}. \tag{13}$$

and for the adiabatic perturbations $c_s^2 = \frac{\delta p}{\delta\rho}$. For the case of matter dominant epoch one can obtain exact solution for differential equation (12), considering ($c_s = 0$), yields:

$$\omega(x,\tau) = E_1(x)\theta(\tau) + E_2(x)\theta(\tau)\int\frac{d\tau}{\theta^2}. \tag{14}$$

Our aim is to obtain density contrast and peculiar velocity in terms of ω. Density contrast $\delta = \frac{\delta\rho}{\delta}$ can be obtained by dividing eq.(8) to $G^0_{\ 0}$ component of the FRW equation ($\mathcal{H}^9 + \kappa = \frac{8\pi G\rho a^2}{3}$):

$$\delta = \frac{2}{3(\mathcal{H}^2 + \kappa)}(\nabla^2\phi - 3\mathcal{H}\phi' - 3(\mathcal{H}^2 - \kappa)\phi). \tag{15}$$

In what follows, Eqs.(9) and (15) will be our main equations in the following sections. It is seen that these two equations are as function of ϕ which, can be obtained by solving eq.(12). In the next section we find an explicit relation between density contrast and peculiar velocity in the framework of general relativistic perturbation theory in the Newtonian gauge. we restrict ourselves to the case of radiation and matter dominant epochs in the flat universe.

4

4 Density Contrast–Peculiar Velocity Relation

From the recent CMB and Supernova type I experiments, it seems that the curvature of the universe is flat [14] ($\kappa = 0$). In this section according to sequence of the radiation and matter dominant epochs from early universe, we apply Eqs.(9) and (15) for these two regimes.

4.1 Radiation dominant epoch

In the radiation dominant epoch, for the flat universe, according to FRW equations, scale factor evolves like $a = a_0\tau$. In this case eq.(13) reduce to:

$$\theta = \sqrt{\frac{3}{4}\frac{1}{a}}. \tag{16}$$

For simplicity in the calculation we consider structures larger than Hubble radius, so the second term in the right hand side of eq.(12) can be ignored with respect to other terms and $\omega(x, \tau)$ can be obtained form eq.(14). Substituting eq.(16) into eq.(14), the dynamics of ω is obtained:

$$\omega = \frac{1}{(\mathcal{H}^2 - \mathcal{H}')^{4/2}}\left(\frac{D(x, \tau)}{a}\right)', \tag{17}$$

where $D(x, \tau) = [E_1\sin(\nu\tau) + E_2\cos(\nu\tau)]e^{(ik\cdot x)}$ and $\nu = \frac{k}{\sqrt{3}}$. Substituting eq.(17) into eqs.(11), (15) and (9), we obtain following equations for $\phi(x, \tau)$, $\delta(x, \tau)$ and $u_i(x, \tau)$:

$$\phi(x, \tau) = \frac{1}{\tau^3}\left[(\nu\tau\cos(\nu\tau) - \sin(\nu\tau))C_1 - (\nu\tau\sin(\nu\tau) + \cos(\nu\tau))C_6\right]e^{(ik\cdot x)}, \tag{18}$$

$$\begin{aligned}\beta(x, \tau) = &\frac{4}{\tau^3}[([(\nu\tau)^2 - 1]\sin(\nu\tau) + \nu\tau[1 - \frac{1}{2}(\nu\tau)^2]\cos(\nu\tau)C_1 \\ &+ ([1 - (\nu\tau)^2]\cos(\nu\tau) + \nu\tau[1 - \frac{1}{2}(\nu\tau)^2]\sin(\nu\tau))C_2)]e^{(ik\cdot x)},\end{aligned} \tag{19}$$

$$\begin{aligned}\nabla\cdot v = &\frac{a_1k^2}{a^2\tau}([(\nu\tau\cos(\nu\tau) - \cos(\nu\tau))C_1 - (\nu\tau\sin(\nu\tau) + \cos(\nu\tau))C_2]e^{(ik\cdot x)} \\ &- \frac{k^2a_0}{2a^2}[(\nu\cos(\nu\tau) - \nu^2\tau\sin(\nu\tau) - \nu\cos(\nu\tau))C_1 \\ &- (\nu\sin(\nu\tau) + \nu^2\tau\cos(\nu\tau) - \nu\sin(\nu\tau))C_2])e^{(ik\cdot x)}.\end{aligned} \tag{20}$$

According to our consideration for the structures larger than Hubble radius $\nu\tau << 1$, eqs.(19) and (20) are reduced as follows:

$$\nabla\cdot v = -\frac{k^2a_0}{a^5\tau}e^{(ik\cdot x)}C_2, \tag{21}$$

$$\delta = \frac{4}{\tau^3}e^{(ik\cdot x)}C_2. \tag{22}$$

Now one can divide eq.(21) by eq.(22) to obtain the explicit relation between density contrast and the peculiar velocity:

$$\nabla\cdot v = -\frac{3}{4}H\delta(\nu\tau)^2. \tag{23}$$

It is seen from eq.(23) that in the radiation dominant epoch, the relativistic peculiar velocity for a given density contrast is much smaller than what one can expect from the Newtonian formula.

4.2   Matter dominant epoch

In the matter dominant epoch, for $\kappa = 0$, eq.(13) is simplified into this form:

$$\theta = \frac{1}{a}. \tag{24}$$

For the flat universe, the scale factor grows as $a = a_0 \tau^2 / 2$. By substituting eq.(24) in eq.(14), the dynamics of ω as a function of conformal time is obtained:

$$\omega = E_1(x)\tau^3 + E_2(x)\tau^{-2}, \tag{25}$$

where $E_1(x)$ and $E_2(x)$ are arbitrary functions of spatial coordinates. Using ω in the eq.(11), the value of ϕ obtain as follows:

$$\phi(x,\tau) = C_1(x) + \tau^{-7}C_2(x). \tag{26}$$

Substituting eq.(26) into eqs.(9) and (15), δ and u are obtained as functions of conformal time, $C_1(x)$ and $C_2(x)$ as:

$$\delta = \frac{1}{6}\left[(\nabla^2 C_6(x) - 81C_2(x)) - \tau^{-5}(\tau^2\nabla^2 C_2(x) + 18C_2(x))\right], \tag{27}$$

$$u_i(x,\tau) = \frac{1}{a_5}\left(\frac{C_2(x)_{,i}}{\tau^6} - \frac{2}{3}\frac{C_1(x)_{,i}}{\tau}\right). \tag{28}$$

Neglecting the decaying modes, we rewrite eqs.(27) and (28) in the Fourier space:

$$\delta_k = -\frac{1}{6}(k^2\tau^2 + 12)C_1(k), \tag{29}$$

$$\frac{ik \cdot v_i(k)}{a} = \frac{2}{3a_1}\frac{k^2}{\tau}C_1(k). \tag{30}$$

Dividing eq.(30) by eq.(29), one can obtain an explicit relation between density contrast and peculiar velocity in the following form:

$$\frac{ik.v_k}{a} = -\frac{H\delta_k}{1 + \frac{3a^2 H^2}{k^2}} \tag{31}$$

where, H is the Hubble constant in physical time and v_k is the Fourier transformation of the peculiar velocity. By using the definition of scalar potential for the peculiar velocity, $v_k = \frac{ik}{a}\mathcal{W}_k$, eq.(31) can be written:

$$H\delta_k = \frac{k^2}{a^2}\mathcal{W}_k + 3H^2\mathcal{W}_k. \tag{32}$$

In the real space, eq.(32) changes to:

$$\nabla \cdot v = -H\frac{\delta}{b} + 3H^2\mathcal{W} \tag{33}$$

where b is biasing factor and $\mathcal{W} = \int v \cdot dl$ is the scalar potential of velocity field and can be obtained by radial integrating of velocity field. The second term in the right hand side

of eq.(33) is a result of general relativistic corrections. This correction can be estimated by dividing the second term by the first term in the right hand side of eq.(32).

$$\frac{3H^2 W_k}{\frac{k^2}{a^2} W_k} \simeq (\frac{\lambda}{d_H})^2 \qquad (34)$$

where d_H is the size of horizon and $\lambda = \frac{a}{k}$ is the size of structure. Is is seen that for the small scale structures (compare to the size of horizon), the general relativistic correction is negligible. This correction for the structures with $300 Mpc$ extension is about one percent.

5 The Effect of Relativistic Peculiar Velocity on CMB

The inhomogeneities in the universe induce anisotropies in the distribution of relic background of the photons on the CMB. The anisotropy of CMB can be caused by several reasons. One of the main reasons is that the matter which scattered the radiation in our direction had a peculiar velocity with respect to comoving frame when the scattering occurred. Since the peculiar velocity of matter is different at different directions in the sky, this leads to an anisotropy on the CMB. The universe at the last scattering epoch has a redshift of $z \approx 1000$. Since the equality of matter and radiation occurred at $z \approx 5 \times 10^4$, we can consider that at last scattering epoch universe resides in the matter dominant. Using eq.(31) for relativistic peculiar velocity in matter dominant epoch, the relativistic anisotropy due to Doppler effect on CMB is obtained as follows:

$$\frac{\delta T}{T} \simeq v = \frac{\frac{\lambda}{d_H}}{1 + 3(\frac{\lambda}{d_H})^2} \delta \qquad (35)$$

where λ is the size of structure and d_H is the Hubble radius at the decoupling epoch. It is seen from eq.(35) that the temperature perturbation of the CMB due to peculiar velocity in the relativistic framework can be expressed in the term of Newtonian one:

$$(\frac{\delta T}{T})_{Rel} = \frac{(\frac{\delta T}{T})_{New}}{1 + 3(\frac{\theta}{\theta_H})^2} \qquad (36)$$

where $(\frac{\delta T}{T})_{New} = \frac{\lambda}{d_H} \delta$ is the contribution of Newtonian Doppler anisotropy, $\theta = 34''(\Omega h)(\frac{\lambda}{3Mpc})$ is the angular size of structure [8] and $\theta_H \simeq 1°$ is the angular size of Hubble radius. It is seen from eq.(36) that for the large angles, the relativistic contribution of Doppler effect due to the peculiar velocity on the fluctuations of CMB is smaller than the contribution of Newtonian anisotropy.

We compare the relativistic Doppler effect also with Sachs–Wolf effect (which is one of the reasons for the the anisotropy of CMB arising from the variation in the gravitational potential at the last scattering [15]). It is shown that the contribution of this effect on anisotropy of CMB can be given by:

$$\frac{\delta T}{T} = \frac{1}{3}(\frac{\theta}{\theta_H})^2 \delta. \qquad (37)$$

Eqs.(35) and (37) shows that Sachs–wolf effect at large angles ($\theta > \theta_H$) is dominant term on the anisotropies of CMB while for the small angles Doppler term is dominant. Fig.

1 shows the fluctuation of temperature as a function of angular size of the anisotropy for the case of Newtonian and relativistic peculiar velocity and Sachs–wolf effect which normalized to the density contrast. It can be shown that the contribution of relativistic peculiar velocity affect the power spectrum of CMB.

6   Conclusion

In this letter, we have tried to identify the peculiar velocity in general relativistic theory of structure formation. We obtain a relation between the density contrast and the peculiar velocity for the matter and radiation dominant epochs. It has been shown that the relativistic correction is about few percent for structures with extension of hundred mega parsec. The effect of the relativistic peculiar velocity as Doppler effect on the CMB also has been calculated. It was shown that the contribution of relativistic peculiar velocity on the anisotropy of CMB is less than what one expected from the Newtonian theory.

References

[1] Peebles, P. J. E. *The large–Scale Structure of the Universe* (Princeton University Press, Princeton, NJ, 1980).

[2] Bertschinger E., Dekel, A, *APJ*, **336**, L5 (1989).

[3] Branchini, E., Zehavi, I., Plionis, M and Dekel, A, *MNRAS*, **313** 419 (2000).

[4] Mansouri, R., Rahvar, S, *Int. J. Mod. Phys. D*, **11**, 321 (2002); astro-ph/9907041.

[5] Lifshitz, E. M, *Phys. USSR.*, **10**, 116 (1946).

[6] Bardeen, J, *Phys. Rev. D*, **22**, 1882 (1980).

[7] Weinberg, S, *Graviation and Cosmology* (Wiley, New York, 1972).

[8] Padmanabhan, T, *Structure Formation in Universe* (Cambridge University Press, Cambridge, England, 1993).

[9] Bertschinger, E ,in *Porceeding of the Cosmology and Large Scale Structure*, edited by R. Schaefer, J. Silk, M. Spiro and J. Zinn-Justin (Elsevier Science, Amesterdam, 1996), p. 273.

[10] Bertschinger, E (astro-ph/0101009).

[11] Bruni, M., Lyth, D. H, *Phys. Lett. B*, **323**, 118 (1994).

[12] Mukhanov, V. F., Feldman, H. A. and Brandenberger, R. H, *Phys. Rep.*, **215**, 1 (1992).

[13] Stewart, J, *Class. Quantum. Grav.*, **7**, 1169 (1990).

[14] Lange, A. E et. al, *Phys. Rev. D*, **63** (2001); astro-ph/0004389.

[15] Sachs, R. K., Wolfe, A. M, *APJ*, **147**, 73 (1967).

7 Figure Caption

This figure shows the normalized perturbation of temperature to density contrast $(\frac{\delta T}{T})/\delta$ as a function of normalized angular size of structures to the size of horizon $\frac{\theta}{\theta_H}$. Solid line, dashed line and dashed–dot line represent the contribution of Newtonian, Relativistic Doppler effect and Sachs–Wolf effect on the anisotropy of CMB, respectively.